

EQUIMORPHY – THE CASE OF CHAINS

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ABSTRACT. Two structures are said to be *equimorphic* if each embeds in the other. Such structures cannot be expected to be isomorphic, and in this paper we investigate the special case of linear orders, here also called chains. In particular we provide structure results for chains having less than continuum many isomorphism classes of equimorphic chains. We deduce as a corollary that any chain has either a single isomorphism class of equimorphic chains or infinitely many.

Dedicated to James E. Baumgartner, for his warmth and inspirations

1. INTRODUCTION

Two structures are called *equimorphic* (see Fraïssé [F00]) if each embeds in the other. Generally one cannot expect equimorphic structures to necessarily be isomorphic. However the famous Cantor-Bernstein-Schroeder Theorem states that this is the case for structures in a language with pure equality: if there is an injection from one set to another and vice-versa, then there is a bijection between these two sets. The same situation occurs in other structures such as vectors spaces, where embeddings are linear injective maps. But as expected it is not in general the case that equimorphic structures are isomorphic.

In this paper we study the case of the language of one binary predicate interpreted as a linear order, also called chains, and show that the situation here is already quite complex. For example one readily sees that the rationals together with a largest point added is a chain equimorphic to the rationals themselves, but certainly not isomorphic as linear orders. In fact we show that for each cardinal κ there is a chain with exactly κ isomorphism classes of equimorphic chains. We further provide structure results for chains having less than continuum isomorphism classes of equimorphic chains, and deduce as a corollary that any chain has either a single isomorphism class of equimorphic chains or infinitely many.

In [T12], Thomassé conjectures that any countable relational structure has either a single isomorphism class of equimorphic structures, countably many, or else continuum many. We verify this conjecture for the case of chains.

We conclude this section with some basic terminology used in this paper. In general an embedding from a structure \mathcal{A} to a structure \mathcal{B} in the same language is an injective map from \mathcal{A} to \mathcal{B} which preserves the given structure; in the case of linear orders $\mathcal{A} = \langle A; <_A \rangle$ and $\mathcal{B} = \langle B; <_B \rangle$, this mean an injective map $f : A \rightarrow B$ such that $x <_A y$ if and only if $f(x) <_B f(y)$ for all $x, y \in A$.

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We will write \mathcal{A}^* for the reverse order $\mathcal{A}^* = \langle A; >_A \rangle$ obtained by reversing the order, and $\mathcal{A} + \mathcal{B}$ for the linear order obtained by extending the two orderings imposing that every element of A precedes every element of B . Given structures \mathcal{A} and \mathcal{B} , we write $\mathcal{A} \leq \mathcal{B}$ if there is an embedding from \mathcal{A} to \mathcal{B} , and write $\mathcal{A} \equiv \mathcal{B}$ if both $\mathcal{A} \leq \mathcal{B}$ and $\mathcal{B} \leq \mathcal{A}$; in this case we say that \mathcal{A} and \mathcal{B} are *equimorphic*, or that \mathcal{B} is a *sibling* of \mathcal{A} (and vice-versa). Note that Bonato et al. [B11] refer to such a \mathcal{B} as a *twin* if moreover it is not isomorphic to \mathcal{A} . We shall be interested in describing these siblings and counting their number but obviously only up to isomorphism, which we denote by $\text{sib}(\mathcal{A})$. Thus $\text{sib}(\mathcal{A}) = 1$ means that all siblings are isomorphic to \mathcal{A} , or that \mathcal{A} has no twins. Note also that obviously $\text{sib}(\mathcal{A}) = \text{sib}(\mathcal{A}^*)$ for any chain \mathcal{A} . We further write as usual $\mathcal{A} \cong \mathcal{B}$ when the two structures are isomorphic.

Other more standard terminology and notation can be found in the books by Fraïssé [F00] and Rosenstein [R82]. In particular we assume the reader to be generally familiar with the notion of indecomposability and Hausdorff rank of a linear order although we briefly review these notions below.

2. ORDINALS, SUMS OF ORDINALS AND REVERSE ORDINALS

It is easy to see that any ordinal has only one sibling up to isomorphism, that is $\text{sib}(\lambda) = 1$ for any ordinal λ . More generally this is the case for any finite sum of ordinals or reverse ordinals.

Proposition 2.1 (Finite sums of ordinals and reverse ordinals). *If C is a finite sum of ordinals and reverse ordinals, then $\text{sib}(C) = 1$.*

Proof. Let n be the least integer such that C has a decomposition as a sum of n ordinals or reverse ordinals. Choose a decomposition $C := \sum_{i < n} C_i$ minimal in the sense that if $C := \sum_{i < n} C'_i$ is an other decomposition with $C'_i \leq C_i$ for $i < n$ then C'_i is equimorphic to C_i for all $i < n$; this exists since ordinals are well ordered under embeddability.

Now consider any chain $C' \equiv C$. Since $C' \leq C$, C' must be of the form $C' := \sum_{i < n} C'_i$ with $C'_i \leq C_i$ for all $i < n$. Since $C \leq C'$, the same argument yields that $C := \sum_{i < n} C''_i$ with $C''_i \leq C'_i$ for all $i < n$. Since $C'_i \leq C_i$ we have $C''_i \leq C_i$. From the minimality of the decomposition of C , we have $C''_i \equiv C_i$ hence $C'_i \equiv C_i$. This yields $C'_i \simeq C_i$ thus $C' \simeq C$. \square

Note that an ordinal chain for example is rigid, meaning it has no non-trivial automorphisms. On the other hand it has non-trivial embeddings, and this subtle distinction will soon play a role. The situation very much differs with infinite sums of ordinals as the following example shows, and this allows us to easily find chains with any prescribed value of siblings, in particular $\text{sib}(\lambda \cdot \omega^*) (= \text{sib}(\lambda^* \cdot \omega)) = |\lambda|$ for any infinite ordinal λ . This should be later compared with Proposition 3.4 below, where we will see that $\text{sib}(\omega^\alpha \cdot \omega^* + \omega^\beta) = 1$ if $\alpha + 1 \leq \beta$, and $\text{sib}(\sum_{n < \omega} \omega^n + \gamma) = 2^{\aleph_0}$ if γ is any ordinal.

Example 2.2 (Chain with many siblings). *For any infinite ordinal λ ,*

$$\text{sib}(\lambda^* \cdot \omega) = |\lambda|.$$

Proof. Let λ be an ordinal and $|\lambda| = \kappa$. Let β be the smallest ordinal such that $\lambda^* \cdot \omega \equiv \beta^* \cdot \omega$, and for every ordinal $\alpha < \beta$, let $C(\alpha) = \alpha^* + (\beta^* \cdot \omega)$. Then one readily sees that these chains $C(\alpha)$ are a complete list of pairwise non-isomorphic siblings. Thus $\text{sib}(\lambda^* \cdot \omega) = \text{sib}(\beta^* \cdot \omega) = \kappa = |\lambda|$. \square

Hence in particular there are chains with continuum many siblings. In fact there are countable chains with that property, and here are two examples that will be important for the sequel.

Example 2.3 (Countable chain with continuum many siblings). (1) $\text{sib}(\mathbb{Z} \cdot \omega) = 2^{\aleph_0}$.
 (2) $\text{sib}(\mathbb{Q}) = 2^{\aleph_0}$.

Proof. For the first part, let $C = \mathbb{Z} \cdot \omega$. Now for $X \subseteq \omega$ infinite, let $C(X) = \sum_{i \in \omega} \mathbb{Z}^{\chi_i}$, where $\chi_i = 1$ if $i \in X$, and 0 otherwise; this means we replace the copy of \mathbb{Z} by a singleton for any index $i \notin X$. Then clearly $C \equiv C(X)$ for any infinite X , but $C(X) \not\equiv C(Y)$ whenever $X \neq Y$, and thus $\text{sib}(C) = 2^{\aleph_0}$.

But now one has that for any $X \subseteq \omega$, $\overline{C(X)} := C(X) + \mathbb{Q} \equiv \mathbb{Q}$, and again $\overline{C(X)} \not\equiv \overline{C(Y)}$ whenever $X \neq Y$, and thus $\text{sib}(\mathbb{Q}) = 2^{\aleph_0}$. \square \square

This yields the following corollary.

Corollary 2.4. $\text{sib}(C) = 2^{\aleph_0}$ for all non-scattered countable chains.
 Equivalently, if C is a countable chain and $\text{sib}(C) < 2^{\aleph_0}$, then C is scattered.

Example 2.3 also yields the following.

Corollary 2.5. If a chain C is an infinite alternating ω -sequence of infinite ordinals and reverse ordinals, then $\text{sib}(C) \geq 2^{\aleph_0}$.

Similarly if a chain is of the form $C = \sum_{i \in \omega} \kappa_i^*$ (or its reverse) where the κ_i 's form a strictly increasing chain of cardinals (or even ordinals of strictly increasing cardinalities), then $\text{sib}(C) \geq \max\{2^{\aleph_0}, \sup_i \{\kappa_i\}\}$.

We will show in Proposition 3.2 that $\text{sib}(C) \geq 2^{\aleph_0}$ whenever $(\omega^* + \omega) \cdot \omega$ or $(\omega^* + \omega) \cdot \omega^*$ are embeddable in a scattered chain C .

We hastily note that there are uncountable dense chains C such that $\text{sib}(C) = 1$, and one such construction is owed to Dushnik and Miller [DM40] (see also Rosenstein [R82]). Indeed they have constructed, by a clever 2^{\aleph_0} -length diagonalization, a dense uncountable subchain C of the real numbers which is *embedding rigid*, meaning that the identity map is the only embedding of C into itself. Clearly this implies that $\text{sib}(C) = 1$.

We note that Baumgartner showed in [B76] that there are κ -dense rigid chains of size κ for each regular and uncountable cardinal κ , although we do not know if these can be made embedding rigid. Hence we ask the following.

Problem 2.6. Are there κ -dense embedding rigid chains of size κ for each regular and uncountable cardinal κ ?

3. STRUCTURE RESULTS

In this section we describe two results characterizing the structure of a chain and its number of siblings, which together yield the following dichotomy:

Theorem 3.1. Let C be any chain. Then $\text{sib}(C) = 1$ or $\text{sib}(C) \geq \aleph_0$.

We will extend Corollary 2.5 as follows:

Proposition 3.2. *If $(\omega^* + \omega) \cdot \omega$ or $(\omega^* + \omega) \cdot \omega^*$ are embeddable in a scattered chain C , then $\text{sib}(C) \geq 2^{\aleph_0}$.*

Chains in which neither $(\omega^* + \omega) \cdot \omega$ nor $(\omega^* + \omega) \cdot \omega^*$ are embeddable have a special form described in Proposition 3.3 below. For this let us first recall the notion of “surordinal” introduced independently by Slater [S64] and Jullien [J67b]. A chain C or its order type is a *surordinal* if for each $x \in C$ the cofinal segment generated by x is well ordered. Equivalently, $1 + \omega^*$ does not embed into C . Jullien called *pure* surordinals those which are strictly left indecomposable (in particular, ordinals are not pure). He observed that every non-pure surordinal is the sum of a pure surordinal and an ordinal; the decomposition is not unique but the pure surordinals figuring in two decompositions are equimorphic; we will call *component* such pure surordinal defined up to equimorphy. Jullien showed in his thesis that a surordinal is pure if and only if it can be written as a sum $\sum_{n < \omega}^* C_n$ where each C_n has order type ω^{α_n} and the sequence $(\alpha_n)_{n < \omega}$ is non-decreasing. Furthermore, this sum is unique (see Jullien [J68c] Proposition 3.3.2).

Proposition 3.3. *Neither $(\omega^* + \omega) \cdot \omega$ nor $(\omega^* + \omega) \cdot \omega^*$ are embeddable into a chain C if and only if C is a finite sum of surordinals and reverse of surordinals.*

Thus, according to Propositions 3.2 and 3.3, scattered chains with few ($< 2^{\aleph_0}$) siblings are finite sums of surordinals and their reverse. In the next proposition, we compute the number of siblings of a surordinal.

Proposition 3.4. *Let C be a surordinal. Then:*

- (1) *$\text{sib}(C) = 1$ if and only if either C is an ordinal, ω^* , or C is not pure but the sequence in a component is stationary, that is $C = \omega^\alpha \cdot \omega^* + \omega^\beta + \gamma$ with $\alpha + 1 \leq \beta$ and γ ordinal.*
- (2) *$\text{sib}(C) = |C|$ if C is pure and the sequence $(\alpha_n)_{n < \omega}$ in the decomposition of C is stationary.*
- (3) *$\text{sib}(C) = |C'|^{\aleph_0}$ if the sequence in a component C' of C is non-stationary.*

The following result describe the scattered chains with few siblings.

Theorem 3.5. *Let C be any chain and $\kappa < 2^{\aleph_0}$. Then the following are equivalent:*

- (1) *$\text{sib}(C) = \kappa$ and C is scattered;*
- (2) *$\kappa = 1$, or $\kappa \geq \aleph_0$ and C is a finite sum of surordinals and of reverse of surordinals, and if $C = \sum_{j < m} D_j$ is such a sum with m minimum then $\max\{\text{sib}(D_j) : j < m\} = \kappa$.*

This immediately yields the following interesting corollary since, by Corollary 2.4, countable chains C such that $\text{sib}(C) < 2^{\aleph_0}$ must be scattered.

Corollary 3.6. *When C is countable, then $\text{sib}(C) = 1$, \aleph_0 , or 2^{\aleph_0} .*

With an emphasis on the indecomposable components of C , we can also prove a more general structure result when the number of siblings is less than the continuum.

Corollary 3.7. *Let C be a chain. Then:*

- (1) *C is scattered and $\text{sib}(C) = \kappa < 2^{\aleph_0}$ if and only if C is a finite sum $\sum_{i < n} C_i$ of ordinals, surordinals of the form $\omega^\alpha \cdot \omega^* + \omega^\beta$ with $\alpha + 1 \leq \beta$, surordinals of the form $\omega^\alpha \cdot \omega^*$ and reverse of such chains.*

Furthermore if the number of components C_i of this sum such that C_i or its reverse is of the form $\omega^\alpha \cdot \omega^*$ with $\alpha \geq 1$ is minimum, then κ is the maximum cardinality of these components.

- (2) $\text{sib}(C)$ is finite and C is scattered if and only if C is a finite sum of ordinals, surordinals of the form $\omega^\alpha \cdot \omega^* + \omega^\beta$ with $\alpha + 1 \leq \beta$, and their reverse. In which case, $\text{sib}(C) = 1$.

We can also prove a more general structure result when the number of siblings is less than the continuum.

Theorem 3.8. *Let C be any chain and $\kappa < 2^{\aleph_0}$. Then the following are equivalent:*

- (1) $\text{sib}(C) = \kappa$.
- (2) $C = \sum_{i \in D} C_i$, where:
 - D is dense (singleton or infinite),
 - each C_i is scattered,
 - $\text{sib}(C_i) = 1$ for all but finitely many $i \in D$,
 - $\max\{\text{sib}(C_i) : i \in D\} = \kappa$, and
 - every embedding $f : C \rightarrow C$ preserves each C_i .

Theorems 3.5 and 3.8 immediately prove Theorem 3.1.

of Theorem 3.1. If C is a chain such that $\text{sib}(C) = \kappa < \aleph_0$, then writing $C = \sum_{i \in D} C_i$ as in Theorem 3.8, we must have $\text{sib}(C_i) = 1$ for each $i \in D$ by Theorem 3.5 since each C_i is scattered, and thus $\kappa = 1 = \text{sib}(C)$ from Theorem 3.8. $\square \square$

We also remark that the dense set D in Theorem 3.8 does not have to be embedding rigid even when $\text{sib}(C) = 1$; in fact even $D = \mathbb{R}$ is possible in that case. Indeed Dushik and Miller [DM40] (see also Rosenstein [R82]) showed that \mathbb{R} can be decomposed into two disjoint dense subsets E and F such that $g(E) \cap F \neq \emptyset$ and $g(F) \cap E \neq \emptyset$ for any non-identity order preserving map $g : \mathbb{R} \rightarrow \mathbb{R}$. Thus if $C = \sum_{i \in \mathbb{R}} C_i$, where:

$$\begin{cases} |C_i| = 2 & \text{if } i \in E \\ |C_i| = 1 & \text{if } i \notin E \text{ (} i \in F \text{)}, \end{cases}$$

then C itself is embedding rigid. This is because given any order preserving map $f : C \rightarrow C$, define a function $\phi : \mathbb{R} \rightarrow \mathcal{P}(\mathbb{R})$ such that $\phi(i) = \{j \in \mathbb{R} : f(C_i) \cap C_j \neq \emptyset\}$. Hence we can define $\overline{\phi(i)}$ as the interval of \mathbb{R} determined by $\phi(i)$. Now we may define an order preserving map $g : \mathbb{R} \rightarrow \mathbb{R}$ by:

$$i \rightarrow \begin{cases} j & \text{if } f(C_i) \subseteq C_j \\ \text{arbitrary } j \in E \cap \overline{\phi(i)} & \text{otherwise.} \end{cases}$$

But then $g(E) \subseteq E$ and hence g is the identity map by assumption, and this immediately implies that f is the identity as well.

On the other hand we can show that the dense set D in Theorem 3.8 cannot be countably infinite.

Proposition 3.9. *If $C = \sum_{i \in D} C_i$ where each C_i is scattered and D is a countably infinite dense chain, then $\text{sib}(C) \geq 2^{\aleph_0}$.*

The proofs will be completed in the next section, but we stress that there are many immediate unanswered questions.

Problem 3.10. *Suppose that $C = \sum_{i \in D} C_i$, where:*

- D is embedding rigid,
- each C_i is scattered,
- $\text{sib}(C_i) = 1$ for all but finitely many $i \in D$, and
- $\max\{\text{sib}(C_i) : i \in D\} = \kappa$.

Does it follow that $\text{sib}(C) = \kappa$?

Another intriguing question is the following.

Problem 3.11. Suppose that a chain C satisfies $\text{sib}(C) = \kappa < 2^{\aleph_0}$, can C be in fact be written as in Problem 3.10?

And further regarding embedding rigidity, we cannot answer the following question in full generality.

Problem 3.12. Suppose that $C = \sum_{i \in D} C_i$, where D and every C_i are embedding rigid, is C necessarily embedding rigid?

The answer here is clearly yes if C is countable as this immediately implies, since D is embedding rigid, that D is in fact finite, and thus each C_i must be finite as well. Similarly if each C_i is countable, then they must be finite. And again the answer is positive if all the C_i are isomorphic.

4. PROOFS

In this section we will prove Propositions 3.2, 3.3, 3.4 and 3.9, and Theorems 3.5 and 3.8.

Thus let C be a chain. By Hausdorff's condensation arguments (see Rosenstein [R82]) we can immediately write $C = \sum_{i \in D} C_i$ where D is dense (singleton or infinite) and each C_i is scattered. To see this, define, for $x, y \in C$, the equivalence relation $x \equiv_0 y$ if the interval $[x, y]$ is finite. Now for successor ordinals, define $x \equiv_{\alpha+1} y$ if the interval $[x/\equiv_\alpha, y/\equiv_\alpha]$ is finite in C/\equiv_α . For a limit ordinal β , simply let $\equiv_\beta := \bigcup_{\alpha < \beta} \equiv_\alpha$. Then the Hausdorff rank of C , written $h(C)$, is the least ordinal α such that $\equiv_\alpha = \equiv_{\alpha+1}$. Then D above is $C/\equiv_{h(C)}$ and the C_i above are simply the $\equiv_{h(C)}$ equivalence classes.

Proof. (of Propositions 3.3)

Suppose that neither $(\omega^* + \omega) \cdot \omega$ nor $(\omega^* + \omega) \cdot \omega^*$ are embeddable into C and that every chain C' with this property and smaller Hausdorff rank is a finite sum of surordinals and reverse surordinals. Let $\alpha = h(C)$. If $\alpha = 0$ then C is either an integer, ω , ω^* or $\omega^* + \omega$, hence an ordinal, the reverse of an ordinal or a surordinal. Suppose $\alpha \geq 1$ and we proceed in two cases.

Case 1. α is a successor ordinal, $\alpha = \alpha' + 1$. Then $D_{\alpha'} = C/\equiv_{\alpha'}$ is either an integer ($\neq 1$), ω , ω^* or $\omega^* + \omega$. By the induction hypothesis, each equivalence class $C_{n, \alpha'}$ of $\equiv_{\alpha'}$ is a finite sum of surordinals and reverse surordinals. Since $C = \sum_{n \in D_{\alpha'}} C_{n, \alpha'}$, if $D_{\alpha'}$ is finite then C is a finite sum of surordinals and reverse surordinals too. If $D_{\alpha'} = \omega$ then for n large enough, either each $C_{n, \alpha'}$ is well ordered, or reversely well ordered, otherwise $(\omega^* + \omega) \cdot \omega$ will be embeddable into C , hence C is a finite sum of surordinals and reverse surordinals. The same argument leads to the same conclusion if $D_{\alpha'}$ is equal to ω^* or to $\omega^* + \omega$.

Case 2. α is a limit ordinal. In this case, every pair of elements x, y of C belongs to some $\equiv_{\alpha'}$ -equivalence class for some $\alpha' < \alpha$. We may write $C = A + B$ with $A = \sum_{\alpha < \kappa}^* A_\alpha$, $B = \sum_{\beta < \lambda} B_\beta$, where κ and λ are cardinals equal respectively to

the cointinality and the cofinality of C and the A_α 's and B_β 's included in some $\equiv_{\alpha'}$ -equivalence classes. If $\lambda \geq \omega$ then for β large enough, say $\beta \geq \beta_0$, B_β is well ordered, or reversely well ordered (otherwise again $(\omega^* + \omega) \cdot \omega$ would be embeddable into C). Since by induction each B_β is a finite sum of surordinals and reverse surordinals, $\sum_{\beta \geq \beta_0} B_\beta$ is such a sum as well. Since via the induction hypothesis $\sum_{\beta' < \beta_0} B_{\beta'}$ is a finite sum of surordinals and reverse surordinals, B is thus such a sum. The same argument applied to A ensures that C is a sum of surordinals and reverse of surordinals.

Conversely, suppose that C is a finite sum of surordinals and reverse of surordinals. If $(\omega^* + \omega) \cdot \omega$ was embeddable into C , then since it is indecomposable, it would be embeddable into a member of this sum, which is clearly impossible. The same argument applies to $(\omega^* + \omega) \cdot \omega^*$. With that the proof is complete. \square \square

But the goal here is to have a more specific structure decomposition. To do so we shall make use of labellings of a chain C by a well quasi-ordered set Q , that is a reflexive and transitive binary relation such that every infinite sequence contains an infinite increasing subsequence. A labelling of C is a pair of the form (C, ℓ) (or simply ℓ when C is clear) where $\ell : C \rightarrow Q$ is an order preserving map. A Q embedding of a labelling (C, ℓ) into another labelling (C', ℓ') is an embedding $f : C \rightarrow C'$ such that $\ell(x) \leq \ell'(f(x))$ for all $x \in C$. Two labellings ℓ and ℓ' (or (C, ℓ) and (C, ℓ')) are said to be isomorphic if there is an automorphism ϕ of C such that $\ell' \circ \phi = \ell$.

We begin with a counting argument for labellings.

Lemma 4.1. *Let C be an infinite chain and $|Q| > 1$. Then there are at least 2^{\aleph_0} pairwise non-isomorphic labellings.*

Proof. Observe that if $|C| = \mu \geq \aleph_0$, $|Q| = \kappa > 1$, and $|Aut(C)| < \kappa^\mu$, then there are $\kappa^\mu \geq 2^{\aleph_0}$ non-isomorphic labellings.

In general write $C = \sum_{i \in C/\equiv_0} F_i$, and thus each F_i is either finite, or type ω , ω^* , or $\omega^* + \omega$. We proceed in cases.

We first consider the case where some F_i is infinite. Then selecting an arbitrary $q \in Q$, we extend each labelling $\ell : F_i \rightarrow Q$ to the labelling $\bar{\ell} : C \rightarrow Q$ by setting $\bar{\ell}(x) = \ell(x)$ for $x \in F_i$, and $\bar{\ell}(x) = q$ otherwise. If now two such labellings $\bar{\ell}$ and $\bar{\ell}'$ are isomorphic via some automorphism ϕ of C , for example $\bar{\ell}' \circ \phi = \bar{\ell}$, then $\phi(F_i) = F_j$ for some j . If $j \neq i$, then ℓ' and ℓ are equal to q on F_i ; if $j = i$, then ϕ induces an automorphism of F_i and ℓ' and ℓ are isomorphic. Hence there are as many isomorphic labellings of C as of F_i , and thus at least 2^{\aleph_0} .

In the case all the F_i are finite, then C/\equiv_0 is infinite and dense, and thus C contains a copy of every countable chain. But each such copy yields a labelling of C into two colours, and since isomorphic labellings yield isomorphic copies and there are 2^{\aleph_0} non-isomorphic countable chains, then there are 2^{\aleph_0} non-isomorphic labellings of C . \square \square

In particular, if C is equal to ω , ω^* or to $\omega^* + \omega$, then there are $|Q|^{\aleph_0}$ non-isomorphic labellings.

Problem 4.2. *Are there generally in fact $2^{|C|}$ non-isomorphic labellings?*

We will also need the following.

Lemma 4.3. *Let C be a chain, α an ordinal and κ a cardinal.*

Then if $\mathcal{M} = \{E \in C / \equiv_\alpha : \text{sib}(E) \geq \kappa\}$ and $\mu = |\mathcal{M}|$, then $\text{sib}(C) \geq \min\{2^{\aleph_0}, \kappa^\mu\}$.

Proof. For $E \in \mathcal{M}$, let $S(E)$ be a collection of κ pairwise non-isomorphic chains equimorphic to E . Then for $\zeta \in \prod_{E \in \mathcal{M}} S(E)$, define

$$C(\zeta) = \sum_{E \in C / \equiv_\alpha} C_{E, \zeta}$$

where

$$C_{E, \zeta} = \begin{cases} \zeta(E) & \text{if } E \in \mathcal{M}, \\ E & \text{otherwise.} \end{cases}$$

Clearly $C(\zeta) \equiv C$ for each ζ . Now for $\zeta, \xi \in \prod_{E \in \mathcal{M}} S(E)$, an isomorphism between $C(\zeta)$ and $C(\xi)$ would preserve \equiv_α classes, and thus would induce an isomorphism g of C / \equiv_α onto C / \equiv_α such that $E \equiv g(E)$ for each $E \in C / \equiv_\alpha$. Clearly $E \in \mathcal{M}$ if and only if $g(E) \in \mathcal{M}$, and thus $\zeta(E) = \xi(g(E))$. This means that the labelled chains (\mathcal{M}, ζ) and (\mathcal{M}, ξ) are isomorphic.

In the case that \mathcal{M} is finite then we must have $\zeta = \xi$, and the number of non-isomorphic labellings is at least

$$\prod_{E \in \mathcal{M}} |S(E)| = \max\{|S(E)| : E \in \mathcal{M}\} \geq \kappa.$$

If on the other hand \mathcal{M} is infinite, then according to Lemma 4.1 there are at least 2^{\aleph_0} non-isomorphic labellings into a set of size at least 2. The conclusion follows. \square \square

Lemma 4.4. *If $f : C \rightarrow C$ is an order preserving map, and for some $x \in C$ the interval determined by x and $f(x)$ is non-scattered, then $\text{sib}(C) \geq 2^{\aleph_0}$.*

Proof. We may assume without loss of generality that $x < f(x)$, and define

$$A = (-\infty, x], \quad M = (x, f(x)), \quad \text{and} \quad B = [f(x), +\infty)$$

Thus f is a witness to $C \leq A + B$. But then $A + X + B \equiv C$ whenever $X \leq M$, since for any such X we have:

$$A + X + B \leq C \leq A + B \leq A + X + B.$$

But now, if $\{S_\alpha : \alpha < 2^{\aleph_0}\}$ is a family of pairwise non-isomorphic countable scattered sets such that $\text{sib}(S_\alpha) > 1$, then let $X_\alpha = \mathbb{Q} + S_\alpha + \mathbb{Q}$ and finally $C_\alpha = A + X_\alpha + B$. From the above remark we immediately have that $C_\alpha \equiv C$ for each α .

Hence $\text{sib}(C) \geq 2^{\aleph_0}$ provided that the C_α 's are pairwise non-isomorphic. Suppose on the contrary that $g : C_\alpha \rightarrow C_\beta$ is an isomorphism for some $\alpha \neq \beta$. Write $C_\alpha = \sum_{i \in D_\alpha} C_{\alpha, i}$ and $C_\beta = \sum_{i \in D_\beta} C_{\beta, i}$, where each $C_{\alpha, i}$ and $C_{\beta, i}$ is scattered and D_α and D_β are dense.

We claim that $\{i \in D_\alpha : \text{sib}(C_{\alpha, i}) > 1\}$ is infinite. Indeed, g carries each $C_{\alpha, i}$ to some $C_{\beta, j}$.

Since S_α is some $C_{\alpha, i}$, S_β some $C_{\beta, j}$ and $S_\alpha \not\equiv S_\beta$, the image of S_α is either included into A or into B . Without loss of generality, we can assume that this image is included into B ; from which it follows that g is an embedding of B into itself. Consider an arbitrary element $x_0 \in S_\alpha$, recursively define $x_{n+1} = g(x_n)$ for $n < \omega$. Without loss of generality we may assume that $x_0 < x_1 = g(x_0)$, and thus observe that g is an embedding of B into itself. Now for each n choose i_n such that

$x_n \in C_{\alpha, i_n}$, but then $C_{\alpha, i_n} \cong S_\alpha$, and thus $\text{sib}(C_{\alpha, i_n}) \geq 2$. Applying Lemma 4.3 with $\alpha = h(C)$, we get $\text{sib}(C_\alpha) \geq 2^{\aleph_0}$, and hence $\text{sib}(C) \geq 2^{\aleph_0}$. \square \square

We will need two other notions of condensation. Let C be a chain, define for $x, y \in C$ the equivalence relation $x \equiv_{\text{well}} y$ (resp. $x \equiv_{\text{well}^*} y$) if the interval $[x, y]$ is well ordered (resp. reversely well ordered), see Rosenstein [R82] pages 72 and 73 for an illustration of these notions.

We need an easy observation whose verification is left to the reader.

Lemma 4.5. *Equivalence classes of \equiv_{well} (resp. \equiv_{well^*}) are the maximal intervals of C which are surordinals (resp. reverse surordinals).*

Furthermore, the intersection of \equiv_{well} and \equiv_{well^} is equal to \equiv_0 .*

Lemma 4.6. *Let A, B two intervals of a chain C , one being a \equiv_{well} -equivalence class and the other an \equiv_{well^*} -class. If $A \cap B$, $A \setminus B$ and $B \setminus A$ are each non-empty, then $A \cap B$ is either finite or has order type $\omega^* + \omega$.*

Proof. From Lemma 4.5 above, $A \cap B$ is a \equiv_0 -class. Hence it is either finite or has type ω^* , ω or $\omega^* + \omega$. We claim that the types ω^* and ω do not occur. Indeed, suppose that $A \cap B$ has type ω (the case ω^* is similar). Let $a \in A \setminus B$ and $b \in B \setminus A$. Without loss of generality we may suppose $a < b$ (otherwise, exchange the names of A and B). Let c be the least element of $A \cap B$. Since the interval $[a, b]$ contains a chain of type ω it is not dually well ordered, hence B is a \equiv_{well} -equivalence class (and since it has a least element it forms a well ordered chain). Now, A must be a \equiv_{well^*} -equivalence class, but this is impossible. Indeed, otherwise $[a, c]$ is dually well ordered, hence it contains a lower cover c' of c ; but this lower cover satisfies $c' \equiv_0 c$ and $c' \notin A \cap B$, contradicting the fact that $A \cap B$ is a \equiv_0 -class. \square \square

Lemma 4.7. *For each \equiv_{well} -equivalence class or \equiv_{well^*} -equivalence class E of a chain C , we have $\text{sib}(C) \geq \text{sib}(E)$.*

The proof follows the same lines of the proof of Lemma 4.3.

We recall the notions of indecomposability (from Rosenstein [R82] and Fraïssé [F00]). A chain C is (additively) *indecomposable* if for every decomposition of C into an initial segment A and a final segment B , C is embeddable either into A or into B ; it is *left indecomposable* if it is embeddable into every non-empty initial segment and it is *strictly left indecomposable* if for every decomposition into a non-empty initial interval A and a final interval B , C is embeddable into A but not into B . Right and strictly right indecomposability are defined in the same way. We recall that if C is indecomposable (resp. strictly right or left indecomposable) and $C' \equiv C$ then C' is indecomposable (resp. strictly right and left indecomposable). We also recall that indecomposable ordinals coincide with ordinals of the form ω^α ; that scattered indecomposable chains are either strictly left indecomposable or strictly right indecomposable and that every scattered chain is a finite sum of indecomposable chains, a quite non-trivial result of Laver [L73].

We will need the following result of Jullien [J68b].

Lemma 4.8. *Let C be a chain. Suppose that C is embeddable in each non-empty final segment of C . If C is not an ordinal then $\text{sib}(C) \geq \aleph_0$.*

Proof. First for an arbitrary chain D , let I_D be the largest well ordered initial segment of D and let $D' := D \setminus I_D$.

Assuming that the chain C is not an ordinal, and thus C' is non-empty, then we claim that $C_n = n + C'$ is equimorphic to C for each $n \in \omega$, and that the C_n 's are pairwise non-isomorphic, thus proving our claim.

First the C_n are not isomorphic since $I_{C_n} = n$. Further, from the fact that C is embeddable in each non-empty final segment, then C immediately embeds in $n + C'$, and moreover since C' is infinite then C (and thus $n + C'$) is embeddable in C for each n . \square \square

We now describe siblings of a finite sum of indecomposable order types.

Lemma 4.9. *Let α be an order type which is a finite sum of indecomposable order types. If $\alpha' \equiv \alpha$, then α' is a finite sum of indecomposable order types.*

Let n , resp n' , be minimal such that $\alpha = \alpha_0 + \dots + \alpha_{n-1}$ (resp. $\alpha' = \alpha'_0 + \dots + \alpha'_{n'-1}$) where each α_i (resp. α'_i) is indecomposable. Then $n' = n$ and $\alpha'_i \equiv \alpha_i$ for $i < n$.

For reader's convenience we give a proof. We record the proof given in Pouzet [P70] (see Proposition II-5.7).

Proof. We say that a decomposition of $\alpha = \alpha_0 + \dots + \alpha_{n-1}$, where each α_i is indecomposable, is *minimal* if $\alpha_i + \alpha_{i+1}$ is not indecomposable for $i+1 < n$. Clearly, a decomposition of minimal length is minimal (our proof will show in particular that the converse holds).

Let $\alpha = \alpha_0 + \dots + \alpha_{n-1}$ be a minimal decomposition and $\alpha' \equiv \alpha$. Let $A' \subseteq \alpha$ with A' of type α' . Set $A'_i = A' \cap \alpha_i$ for $i < n$. Clearly, $A' = \sum_{i < n} A'_i$. We claim that $A'_i \equiv \alpha_i$ for every $i < n$, proving that the A'_i 's form a decomposition of A' . Indeed, since $\alpha \leq \alpha'$ there is an embedding f of α into A' . Let $i < n$; since α_i is indecomposable, there is some $k < n$ such that $\alpha_i \equiv \alpha_i \cap f^{-1}(A'_k)$; let $\varphi(i)$ be the least k with this property. This allows to define a map φ from n into n . This map being order preserving then, as observed by Jullien [J68c] Lemma 3.4.1, if it is not the identity there is some $i < n$ such that either $i = \varphi(i) = \varphi(i+1)$ or $i = \varphi(i-1) = \varphi(i)$ (the first case occurs if there is $x < f(x)$ and the second case if there is some $x > f(x)$).

In the first case $\alpha_i + \alpha_{i+1} \leq A'_i \leq \alpha_i$, proving that $\alpha_i + \alpha_{i+1}$ is indecomposable and contradicting the minimality of the decomposition of α ; the second case is similar.

Thus φ is the identity map. Consequently $A'_i \equiv \alpha_i$ for all i as claimed. We may observe furthermore that the decomposition of A' induced by the decomposition of α is minimal. Let $\alpha' = \alpha'_0 + \dots + \alpha'_{n'-1}$ be a minimal decomposition. Without loss of generality, we may suppose $n' \geq n$. This decomposition of α' induces a decomposition of A' as $A' = \sum_{i' < n'} A''_{i'}$. As above, we may define φ' from n' into n , setting $\varphi(i') = k$ where k is minimum such that $A''_{i'} \equiv A'_k$ and $A'_k \cap A''_{i'} \neq \emptyset$. Due to the minimality of the decomposition of α' , φ' is one to one, hence the identity, hence $n' = n$ and $\alpha'_i \equiv \alpha_i$ for $i < n$ as required. \square \square

We are now ready to start counting the number of siblings of finite sum of indecomposable order types.

Lemma 4.10. *Let α be an order type which is a finite sum of indecomposable order types and let $\alpha = \alpha_0 + \dots + \alpha_{n-1}$ be a minimal decomposition.*

Then for each $i < n-1$, $\text{sib}(\alpha) \leq \text{sib}(\alpha_0 + \dots + \alpha_i) \times \text{sib}(\alpha_{i+1} + \dots + \alpha_{n-1})$.

Furthermore, equality holds if α_i is strictly right indecomposable or α_{i+1} is strictly left indecomposable.

Proof. Set $\bar{\alpha}_i := \alpha_0 + \dots + \alpha_i$ and $\underline{\alpha}_{i+1} := \alpha_{i+1} + \dots + \alpha_{n-1}$. Let $(\bar{\beta}'_i, \underline{\beta}'_{i+1}) \in \text{sib}(\bar{\alpha}_i) \times \text{sib}(\underline{\alpha}_{i+1})$ and $\vartheta(\bar{\beta}'_i, \underline{\beta}'_{i+1}) = \bar{\beta}'_i + \underline{\beta}'_{i+1}$. Clearly, ϑ maps $\text{sib}(\bar{\alpha}_i) \times \text{sib}(\underline{\alpha}_{i+1})$ into $\text{sib}(\alpha)$. We claim that this maps is surjective. Indeed, let $\alpha' \equiv \alpha$, then according to Lemma 4.9, $\alpha' = \alpha'_0 + \dots + \alpha'_{n-1}$ with $\alpha'_i \equiv \alpha_i$ for $i < n$. Setting $\bar{\alpha}'_i := \alpha'_0 + \dots + \alpha'_i$ and $\underline{\alpha}'_{i+1} := \alpha'_{i+1} + \dots + \alpha'_{n-1}$, we have $(\bar{\alpha}'_i, \underline{\alpha}'_{i+1}) \in \text{sib}(\bar{\alpha}_i) \times \text{sib}(\underline{\alpha}_{i+1})$ and $\vartheta(\bar{\alpha}'_i, \underline{\alpha}'_{i+1}) = \alpha'$. This proves the surjectivity; the inequality between cardinals follows.

Now we prove that this map is one to one provided that α_i is strictly right indecomposable or α_{i+1} is strictly left indecomposable. We suppose that α_i is strictly right indecomposable (the case α_{i+1} strictly left indecomposable is similar). We prove that if $\alpha' \in \text{sib}(\alpha)$ there is a unique pair $(\bar{\alpha}'_i, \underline{\alpha}'_{i+1}) \in \text{sib}(\bar{\alpha}_i) \times \text{sib}(\underline{\alpha}_{i+1})$ such that $\vartheta(\bar{\alpha}'_i, \underline{\alpha}'_{i+1}) = \alpha'$. The existence of such a pair was proved above. If $(\bar{\alpha}''_i, \underline{\alpha}''_{i+1})$ is another pair, then either $\bar{\alpha}''_i$ is an initial segment of $\bar{\alpha}'_i$ or $\bar{\alpha}'_i$ is an initial segment of $\bar{\alpha}''_i$. We may suppose that $\bar{\alpha}''_i$ is an initial segment of $\bar{\alpha}'_i$. We prove that if fact $\bar{\alpha}''_i = \bar{\alpha}'_i$ and thus $\underline{\alpha}''_{i+1} = \underline{\alpha}'_{i+1}$, yielding the injectivity as required. Indeed, since α_i is strictly right indecomposable and $\alpha'_i \equiv \alpha_i$ then α'_i is strictly right indecomposable. If $\bar{\alpha}''_i$ is a proper initial segment of $\bar{\alpha}'_i$, let u such that $\bar{\alpha}'_i = \bar{\alpha}''_i + u$. Then u is a proper final segment of α'_i (otherwise $\bar{\alpha}''_i \leq \alpha'_0 + \dots + \alpha'_{i-1} < \bar{\alpha}''_i$ which is impossible since $\bar{\alpha}''_i \equiv \bar{\alpha}'_i$). Let v such that $\alpha'_i = v + u$; then $\bar{\alpha}''_i = \alpha''_0 + \dots + \alpha''_{i-1} + v$. Since this is a minimal decomposition of $\bar{\alpha}_i$ we have $v \in \text{sib}(\alpha_i)$ and since α'_i is strictly right indecomposable, we have $v < \alpha'_i$, a contradiction. \square \square

From this we deduce the following two extensions we will need.

Corollary 4.11. *Let α be an order type which is a finite sum of indecomposable order types and let $\alpha = \alpha_0 + \dots + \alpha_{n-1}$ be a minimal decomposition.*

If α_i is strictly left indecomposable and α_{i+1} is strictly right indecomposable, then

$$\text{sib}(\alpha) = \text{sib}(\alpha_0 + \dots + \alpha_{i-1}) \times \text{sib}(\alpha_i + \alpha_{i+1}) \times \text{sib}(\alpha_{i+2} + \dots + \alpha_{n-1}).$$

Corollary 4.12. *Let α be an order type which is a finite sum of indecomposable order types and let $\alpha = \alpha_0 + \dots + \alpha_{n-1}$ be a minimal decomposition where each α_i is either strictly left or strictly right indecomposable. Now let*

$$K = \{i < n - 1 : \begin{array}{l} \alpha_i \text{ is strictly left indecomposable and} \\ \alpha_{i+1} \text{ is strictly right indecomposable.} \end{array}\}$$

Then,

$$(1) \quad \text{sib}(C) = \prod_{i \notin K} \text{sib}(\alpha_i) \times \prod_{i \in K} \text{sib}(\alpha_i + \alpha_{i+1}).$$

Now let κ be a cardinal. Following Laver in [L73], we call a family $(\alpha_\lambda)_{\lambda < \kappa}$ of order types *unbounded* if for every $\lambda < \kappa$ the set of μ such that $\alpha_\lambda \leq \alpha_\mu$ has cardinality κ . According to Laver [L73], if κ is regular and $(\alpha_\lambda)_{\lambda < \kappa}$ is unbounded then $\sum_{\lambda < \kappa} \alpha_\lambda$ and $\sum_{\lambda < \kappa}^* \alpha_\lambda$ are indecomposable, respectively on the right and on the left.

The following result is essentially in Jullien [J68c] and also Laver [L73].

Lemma 4.13. *Let \mathcal{C} be a collection of order types which is closed downward under embeddability, that is $\alpha \in \mathcal{C}$ and $\beta \leq \alpha$ imply $\beta \in \mathcal{C}$. Then*

- (1) *\mathcal{C} is well quasi ordered under embeddability if and only if the collection $\text{Ind}(\mathcal{C})$ of indecomposable members of \mathcal{C} is well quasi ordered and every member of \mathcal{C} is a finite sum of members of $\text{Ind}(\mathcal{C})$.*

- (2) If \mathcal{C} is well quasi ordered under embeddability, then every unbounded κ sequence $(\alpha_\lambda)_{\lambda < \kappa}$ with κ regular has a final sequence which is unbounded.
- (3) Let α such that for every $x < y \in \alpha$ the interval $[x, y]$ belongs to \mathcal{C} . If \mathcal{C} is well quasi ordered, then α is a finite sum of indecomposable order types $\alpha_0 + \cdots + \alpha_{n-1}$ where:
- $\alpha_i \in \text{Ind}(\mathcal{C})$ for $i \neq 0, n-1$,
 - $\alpha_0 \in \text{Ind}(\mathcal{C})$ or α_0 is a reverse ordinal sum of a regular unbounded sequence of members of $\text{Ind}(\mathcal{C})$,
 - $\alpha_{n-1} \in \text{Ind}(\mathcal{C})$ or α_{n-1} is an ordinal sum of a regular unbounded sequence of members of $\text{Ind}(\mathcal{C})$.

The proof of (1) relies on Higman's theorem on words [H52]. For the proof of (2), set $I(F) := \{\beta \in \mathcal{C} : \beta \leq \alpha_\lambda \text{ for some } \lambda \in F\}$ for each final segment F of κ . Since \mathcal{C} is well quasi ordered, the set of its initial segments is well founded (Higman [H52]). Let F_0 such that $I(F)$ is minimal. Then $(\alpha_\lambda)_{\lambda \in F_0}$ is unbounded. The proof of (3) follows from (1) and (2).

We will use the following consequence of Lemma 4.13. Let S be the class of surordinals. As proved directly by Jullien [J67b], S is well quasi ordered, hence the class U of surordinals and their dual is well quasi ordered. Thus (by (1) of Lemma 4.13), the class $\Sigma(U)$ of finite sums of members of U is well quasi ordered. According to (2) of Lemma 4.13, if each interval $[x, y]$ of a chain C belongs to $\Sigma(U)$, then C is a finite sum $C_0 + \cdots + C_{n-1}$ where $C_i \in \text{Ind}(U)$ for $i \neq 0, n-1$, C_0 and C_{n-1} are a reverse ordinal sum and an ordinal sum of regular unbounded sequences of members of $\text{Ind}(U)$.

A famous theorem of Laver [L73], answering positively Fraïssé's conjectures on chains asserts that the class \mathcal{D} of scattered order types is well quasi ordered under embeddability (see the exposition by Rosenstein in [R82] and Fraïssé in [F00]). From this follows that every scattered order type is a finite sum of indecomposable order types. A consequence is the following Lemma 4.14.

Lemma 4.14. *If a scattered chain C is indecomposable and infinite, there is some equimorphic chain C' such that all the \equiv_0 -equivalence classes are infinite.*

We need a weaker statement, namely the conclusion of this lemma when C is an infinite member of $\text{Ind}(U)$ or an ω -sum of an unbounded sequence of members of $\text{Ind}(U)$. This does not require the well quasi order of \mathcal{D} , which is proved by means of Nash-Williams theory of better quasi ordering, but only the well quasi ordering of $\text{Ind}(U)$. The interest of this weakening could be in the programme of reverse mathematics, see for example Montalbán [M07]. The proof of this weakening is straightforward. Given a chain C , let $F_{\equiv_0}(C)$ be the set of $x \in C$ such that the \equiv_0 -class of x is finite. Note that $F_{\equiv_0}(C)$ is empty if C is a surordinal without a largest element, hence $F_{\equiv_0}(C)$ is empty if C is an infinite member of $\text{Ind}(U)$. Now if C is an ω -sum of an unbounded sequence of members of $\text{Ind}(U)$, say $C = \sum_{n < \omega} C_n$, let A be the set of C_n which are infinite (and in fact have more than one element). If A is finite then it is empty and C is the chain ω , and hence $F_{\equiv_0}(C)$ is empty. If A is infinite, let $C' = \sum_{n \in A} C_n$. Then $F_{\equiv_0}(C')$ is empty; now, as it is easy to see using the unboundedness of the sequence of C_n , C' is equimorphic to C , giving the required conclusion.

We now tackle the proofs of Propositions 3.4 and 3.2 which will lead us to that of Theorem 3.5 on the characterization of scattered sets with a small number of twins.

Proof. (of Proposition 3.4)

We may suppose that C is not an ordinal (otherwise $\text{sib}(C) = 1$). In this case, C decomposes as $C' + D$ where $C' = \sum_{n < \omega}^* C_n$, $(C_n)_{n < \omega}$ is a non-decreasing sequence of ordinals of type ω^{α_n} and D is an ordinal, this ordinal being 0 if C is pure and, otherwise, a non-increasing sequence $D_0 + \dots + D_{n_0}$ of ordinals D_i of type ω^{β_i} with $\alpha_n + 1 \leq \beta_0$ for every n .

If $(C_n)_{n < \omega}$ is stationary and $\alpha = \max\{\alpha_n : n < \omega\}$, then we may rewrite C as $\omega^\alpha \cdot \omega^* + \gamma$, with $\gamma < \omega^{\alpha+1}$ if C is pure, and $\omega^\alpha \cdot \omega^* + \omega^\beta + \gamma$ where $\alpha + 1 \leq \beta$ otherwise. This yields $\text{sib}(C) = |\omega^{\alpha+1}| = |C|$ if C is pure and $\alpha \neq 0$, and $\text{sib}(C) = 1$ otherwise.

If $(C_n)_{n < \omega}$ is non-stationary, then $\text{sib}(C) = |C'|^{\aleph_0}$. Indeed, let $\mu = \sup\{\omega^{\alpha_n} + 1 : n < \omega\}$ and $(C'_n)_{n < \omega}$ be a non-decreasing sequence of ordinals of types $\omega^{\alpha'_n}$ such that $\sup\{\omega^{\alpha'_n} + 1 : n < \omega\} = \mu$. Then $\sum_{n < \omega}^* C'_n$ is equimorphic to $\sum_{n < \omega}^* C_n$ and is isomorphic to this sum if and only if $(C'_n)_{n < \omega} = (C_n)_{n < \omega}$. Since there are at least $|\mu|^{\aleph_0}$ such sequences and $|\mu|^{\aleph_0} = |C'|^{\aleph_0}$, this does the case $D = 0$. If $D \neq 0$, the fact that $\sum_{n < \omega}^* C'_n + D$ is isomorphic to C implies that some final sequence of $(C'_n)_{n < \omega}$ coincide with a final sequence of $(C_n)_{n < \omega}$. This again yields at least $|\mu|^{\aleph_0} = |C'|^{\aleph_0}$ equimorphic types.

This completes the proof of the proposition. \square \square

Proof. (of Proposition 3.2)

We prove this proposition by induction on the Hausdorff rank of the given chain C . Suppose that either $(\omega^* + \omega) \cdot \omega$ or $(\omega^* + \omega) \cdot \omega^*$ are embeddable into C and that $\text{sib}(C') \geq 2^{\aleph_0}$ for every chain C' with this property and smaller Hausdorff rank. Let $\alpha = h(C)$, and note that necessarily $\alpha \geq 1$. Now for $\alpha' < \alpha$, if C' is an $\equiv_{\alpha'}$ -equivalence class, Lemma 4.3 ensures that $\text{sib}(C) \geq \text{sib}(C')$, hence if $\text{sib}(C') \geq 2^{\aleph_0}$ we will have $\text{sib}(C) \geq 2^{\aleph_0}$ as required. This will be the case if either $(\omega^* + \omega) \cdot \omega$ or $(\omega^* + \omega) \cdot \omega^*$ are embeddable into C' . Indeed, the induction hypothesis insures that $\text{sib}(C') \geq 2^{\aleph_0}$. Hence we may suppose that neither $(\omega^* + \omega) \cdot \omega$ nor $(\omega^* + \omega) \cdot \omega^*$ are embeddable into C' and $\text{sib}(C') < 2^{\aleph_0}$ (in fact, with Lemma 4.3 we can suppose that except for finitely many, all $\equiv_{\alpha'}$ -equivalence classes C' satisfy $\text{sib}(C') = 1$).

We claim that we may write C as a finite sum $C = C_0 + \dots + C_{n-1}$ where $C_i \in \text{Ind}(U)$ for $i \neq 0, n-1$, $C_0 \in \text{Ind}(U)$ or C_0 an unbounded ω^* -sum of members of U , namely $C_0 = \sum_{n < \omega}^* A_n$, $C_{n-1} \in \text{Ind}(U)$ or C_{n-1} is an unbounded ω -sum $C_{n-1} = \sum_{m < \omega} B_m$. For that claim, we distinguish two cases.

Case 1: α is a successor ordinal, $\alpha = \alpha' + 1$. In this case, $D_{\alpha'} = C / \equiv_{\alpha'}$ is either an integer ($\neq 1$), ω , ω^* or $\omega^* + \omega$. The chain D cannot be finite (otherwise either $(\omega^* + \omega) \cdot \omega$ or $(\omega^* + \omega) \cdot \omega^*$ would be embeddable into some $\equiv_{\alpha'}$ -equivalence class). Since each $\equiv_{\alpha'}$ -equivalence class belongs to $\Sigma(U)$ we may rewrite C as a sum, indexed by ω , ω^* or $\omega^* + \omega$, of members of $\text{Ind}(U)$. According to (2) of Lemma 4.13, C has a decomposition as above.

Case 2. α is a limit ordinal. In this case, from (2) of Lemma 4.13, C has a decomposition where $C_0 \in \text{Ind}(U)$ or C_0 is a unbounded κ^* -sum of members of $\text{Ind}(U)$, namely, $C_0 = \sum_{\alpha < \kappa}^* A_\alpha$, and $C_{n-1} \in \text{Ind}(U)$ or C_{n-1} is an unbounded λ -sum of members of $\text{Ind}(U)$, namely, $C_{n-1} = \sum_{\beta < \lambda} B_\beta$, with κ, λ regular cardinals.

Necessarily, $\kappa = \lambda = \omega$. Indeed if, for an example $\lambda > \omega$, then since neither $(\omega^* + \omega) \cdot \omega$ nor $(\omega^* + \omega) \cdot \omega^*$ are embeddable into $\sum_{\beta < \beta'} B_\beta$ for $\beta' < \lambda$ we have the same property for C_{n-1} hence $C_{n-1} \in \Sigma(U)$ and refining the decomposition we may suppose $C_{n-1} \in \text{Ind}(U)$. According to the weakening of Lemma 4.14, each infinite C_i is equimorphic to some C'_i such that all the \equiv_0 -equivalence classes are infinite. Thus replacing those C_i by C'_i we get an equimorphic chain C' having only finitely many finite \equiv_0 -equivalence classes, each made of consecutive C'_i of size 1. Let M be maximum of their size. Either C'_0 or C'_{n-1} does not belong to $\text{Ind}(U)$. Suppose $C'_{n-1} \notin \text{Ind}(U)$. Let $\varphi \in 2^{\aleph_0}$ and $E_{\varphi(m)}$ be the chain $\omega + \varphi(m) + M + \omega^*$. Set C'_φ obtained by substituting $C'_{n-1, \varphi} = \sum_{m < \omega} (B'_m + E_{\varphi(m)})$ to $C'_{n-1, \varphi} = \sum_{m < \omega} B'_m$. Since C'_{n-1} is an unbounded sum containing $(\omega^* + \omega) \cdot \omega$, $C'_{n-1, \varphi}$ is equimorphic to C'_{n-1} , hence C'_φ is equimorphic to C' . Furthermore, if $\varphi \neq \varphi'$ then $C'_\varphi \not\cong C'_{\varphi'}$. Hence $\text{sib}(C') \geq 2^{\aleph_0}$ as claimed. \square \square

We now have all the tools to complete the proof of Theorem 3.5.

of Theorem 3.5. (i) \Rightarrow (ii).

Suppose that C is scattered and $\text{sib}(C) = \kappa < 2^{\aleph_0}$. According to Propositions 3.2 and 3.3, C is a finite sum $\sum_{j < m} D_j$ of surordinals and their reverse, and we may suppose m minimum. We prove that in this case $\max\{\text{sib}(D_j) : j < m\} = \text{sib}(C)$, and for this we do not require that $\text{sib}(C) < 2^{\aleph_0}$.

According to Lemma 4.5, each D_j is contained into some equivalence class \overline{D}_j of \equiv_{well} or of \equiv_{well^*} . We can write $\overline{D}_j = U + D_j + V$, and we claim that $\text{sib}(D_j) = \text{sib}(\overline{D}_j)$. Indeed, if U and V are finite, this follows from the computation of the siblings in Proposition 3.4. If U is infinite then since m is minimum, $U = D_{j-1} \cap D_j$, hence $U \subset \overline{D}_{j-1} \cap \overline{D}_j$; furthermore Lemma 4.6 applies and hence $\overline{D}_{j-1} \cap \overline{D}_j$ has order type $\omega^* + \omega$. From this, it follows that \overline{D}_j has order type $\omega^* + \omega + \lambda$ for some ordinal λ and furthermore V is finite (apply Lemma 4.6). Thus in this case $\text{sib}(D_j) = \text{sib}(\overline{D}_j) = 1$. Since $\text{sib}(\overline{D}_j) \leq \text{sib}(C)$ by Lemma 4.7, we have $\text{sib}(D_j) \leq \text{sib}(C)$. Setting $\kappa' = \max\{\text{sib}(D_j), j < m\}$ we obtain $\kappa' \leq \text{sib}(C)$.

We prove that $\kappa' = \kappa$. If $\kappa = 1$, then since $\kappa' \leq \kappa$ the property holds. We may suppose $\kappa > 1$. Since each D_j is a surordinal or the reverse of a surordinal, it has a decomposition as a finite sum of indecomposable chains, hence C has a decomposition as a finite sum $C = \sum_{i < n} C_i$ where the C_i 's are indecomposable, and again we may suppose n minimum. Since these indecomposables are either strictly right or strictly left indecomposable, Formula 1 from Corollary 4.12 applies. Each C_i is a surordinal or the reverse of a surordinal, (indeed since $C_i \subseteq \sum_{j < m} D_j$ and C_i is indecomposable, C_i is embeddable into some D_j ; since D_j is either a surordinal or the reverse of a surordinal, the assertion follows). It follows from Proposition 3.4 that $\text{sib}(C_i)$ and $\text{sib}(C_i + C_{i+1})$ are 1 or infinite, thus $\text{sib}(C) = \max\{\text{sib}(C_i) : i \notin K, \text{sib}(C_i + C_{i+1}) : i \in K\}$. If $\text{sib}(C_i) = \kappa$, let $j < m$ such that $C_i \setminus D_j < C_i$. In this case $\text{sib}(C_i \cap D_j) = \kappa$, and hence $\text{sib}(D_j) = \kappa$. If $i \in K$ and $\text{sib}(C_i + C_{i+1}) = \kappa$, then since $\kappa > 1$, $\kappa = \max\{\text{sib}(C_i), \text{sib}(C_{i+1})\}$. There is some $i < m$ such that $\text{sib}((C_i + C_{i+1}) \cap D_j) = \kappa$ (note that if $C_i \setminus D_j < C_i$, we have $\text{sib}(C_i \cap D_j) = \text{sib}(C_i)$), hence $\text{sib}(D_j) = \kappa$.

(ii) \Rightarrow (i). Since C is a finite sum of surordinal and reverse of surordinals, C is scattered. The fact that $\text{sib}(C) = \kappa$ follows from that fact that $\max\{\text{sib}(D_j), j < m\} = \text{sib}(C)$ proved just above. \square \square

We now provide the argument for Corollary 3.7.

of Corollary 3.7. Note first that part (2) follows from part (1), thus let us prove that (1) holds.

Suppose that C is scattered and $\text{sib}(C) = \kappa < 2^{\aleph_0}$. According to Theorem 2, C is a finite sum $\sum_{j < m} D_j$ of surordinals and their reverse and, m being minimum, $\kappa = \max\{\text{sib}(D_j) : j < m\}$. According to Proposition 3.4, each D_j or its reverse is either an ordinal, a surordinal of the form $\omega^\alpha \cdot \omega^* + \gamma$ with $\gamma < \omega^{\alpha+1}$ if D_j is pure, and $\omega^\alpha \cdot \omega^* + \omega^\beta + \gamma$ with $\alpha + 1 \leq \beta$ otherwise.

We may write $D_j = D_{j,0} + D_{j,1}$ with $D_{j,0} = \omega^\alpha \cdot \omega^*$ and $D_{j,1} = \gamma$ if D_j is pure (and $\alpha \neq 0$), and $D_{j,0} = \omega^\alpha \cdot \omega^* + \omega^\beta$ and $D_{j,1} = \gamma$ if D_j is not pure. From this we obtain a decomposition as stated in the corollary.

In any other decomposition, the number of components of the form $\omega^\alpha \cdot \omega^*$ or its reverse cannot be smaller (otherwise, in order to be eliminated in such other decomposition, a component $D_{j,0} = \omega^\alpha \cdot \omega^*$ will appear in a surordinal of the form $\omega^\alpha \cdot \omega^* + \omega^\beta + \gamma$ with $\alpha + 1 \leq \beta$. Due to the minimality of m , this surordinal must be an initial segment of D_j , which is impossible). Since $\text{sib}(D_{j,0}) = \text{sib}(D_j)$, κ is the maximum of the cardinality of the pure $D_{j,0}$ (distinct from ω^*) and their reverse.

Now, conversely, suppose that C has a decomposition $\sum_{i < n} C_i$ as stated. Then, since the members of this decomposition are surordinals and their reverse, C is scattered and according to Theorem 2, C is a finite sum $\sum_{j < m} D_j$ of surordinals and their reverse and, m being minimum, $\kappa = \max\{\text{sib}(D_j) : j < m\}$. Since $\omega^\alpha \cdot \omega^*$ is indecomposable, each C_i of this form is embeddable into some D_j . If $\alpha \geq 1$, the minimality of n ensures that $D_j = \omega^\alpha \cdot \omega^* + \gamma$ with $\gamma < \omega^{\alpha+1}$, hence $\text{sib}(D_j) = |\omega^\alpha \cdot \omega^*|$. Since for the other D_j 's or their reverse, we have $\text{sib}(D_j) = 1$, we obtain that κ is the maximum of the cardinality of the C_i of the form $\omega^\alpha \cdot \omega^*$ (with $\alpha \geq 1$) or its reverse. \square \square

The next argument proves the full characterization of chains with a small number of siblings.

of Theorem 3.8. We may assume that C is non-scattered, and write $C = \sum_{i \in D} C_i$ where each C_i is scattered and D is dense and infinite.

First if $\text{sib}(C) = \kappa < 2^{\aleph_0}$, then by Lemma 4.4 any embedding must preserve each C_i . But now Lemma 4.3 immediately yields the remaining properties.

Conversely, assume that C has the prescribed decomposition. Then for a given $C' \equiv C$ we may assume that $C' \subseteq C$, and thus we can define $C'_i = C_i \cap C'$ which must then be scattered. Moreover any embedding $f : C \rightarrow C'$ must by assumption satisfy $f(C_i) \subseteq C'_i$ and thus each $C'_i \equiv C_i$. This immediately gives $\text{sib}(C) \leq \kappa$. Since clearly we also have $\text{sib}(C) \geq \kappa$, then $\text{sib}(C) = \kappa$ as desired. \square \square

Finally it remains to complete the proof of Proposition 3.9.

of Proposition 3.9. Let $C = \sum_{i \in D} C_i$ where each C_i is scattered and D is a countably infinite dense chain, we must show that $\text{sib}(C) \geq 2^{\aleph_0}$.

Since D is dense, each C_i is an $\equiv_{h(C)}$ -class and we may apply Lemma 2. If some C_i has 2^{\aleph_0} sibling or infinitely many C_i have more than one siblings then C has 2^{\aleph_0} siblings. Thus we may suppose that each C_i except finitely many have one sibling, those exceptions having less than 2^{\aleph_0} siblings. According to Theorem 2, each C_i is

a finite sum of surordinals and reverse surordinals, thus as proved by Jullien[J67b] the set $Q := \{C_i : i \in D\}$ is well quasi ordered. Let Q be the collection of initial segments of Q , that is $I(Q) = \{I \subseteq Q : C \in I, C' \in Q \text{ and } C' \leq C \Rightarrow C' \in I\}$. Then by a result of Higman, Q is well quasi ordered exactly when $I(Q)$ is well founded. Thus, in our case $I(Q)$ is well founded.

Now for an interval J of D with at least two elements, define:

$$I(J) = \{C_i \in Q : \exists j \in J \text{ such that } C_i \leq C_j\}$$

Since each $I(J)$ is an initial segment of Q , we can choose an interval J so that $I(J)$ is minimal under inclusion among all such collections of the form $I(J')$.

We shall now produce a non-trivial order preserving map $h : J \rightarrow J$ such that $C_x \leq C_{h(x)}$ for all $x \in J$. Since such a map h can be extended to the identity on D outside J , it immediately yields a map f as in by Lemma 4.4, giving $\text{sib}(C) \geq 2^{\aleph_0}$ as desired.

To construct h , first (using the fact that Q is well quasi-ordered) choose $x \neq y \in J$ such that $C_x \leq C_y$ and define $h(x) = y$. Since D and thus J is countable, it suffice to show that given $x \in J$ and h defined on a finite set of J can always be extended to x .

So assume that h is defined on $x_0 < x_1 < \dots < x_{n-1}$, and let $x \in A_i = (x_{i-1}, x_i)$. If $A'_i = (h(x_{i-1}), h(x_i))$, then $I(A_i) = I(J) = I(A'_i)$ by minimality, so there must be $y \in A'_i$ such that $C_x \leq C_y$, and simply define $h(x) = y$.

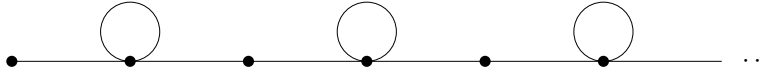
This completes the proof. \square

5. CONCLUSION

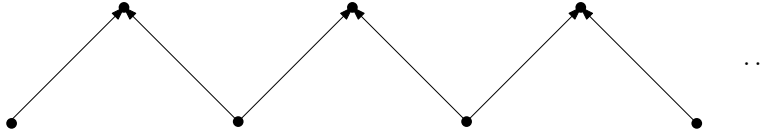
In this paper we studied equimorphy in the natural case of chains, providing some structure results for those having a small number of siblings. Our study was motivated by the following tree alternative conjecture of Bonato and Tardif [BT06]: for every tree T , the number of trees (counted up to isomorphism) which are equimorphic to T , is one or infinite. Partial results were obtained so far by Tyomkim [T09] and extended to graphs by Bonato et al. [B11]. They asked if for every (connected) undirected graph G the number of (connected) graphs (counted up to isomorphism) which are equimorphic to G , is one or infinite. Notice that if one considers connected graphs with loops the conjecture is false. Indeed consider the following undirected graph G with loops.



One can easily verify that in this case $\text{sib}(G) = 2$, with the following graph its only non-isomorphic sibling:



This is also the case for connected posets, as we may simply consider a one way infinite fence, which has two equimorphic siblings:



Thus a complete understanding of the problem not only for trees and posets but for the general case of a relational structure remains very interesting.

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